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FAST TRACK COMMUNICATION

Two new solvable potentials

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Abstract
We succeeded in finding two new solvable potentials by working in a complete square integrable basis that carries a tridiagonal (Jacobi) matrix representation for the wave operator. The conventional methods do not lead to exact solutions for these potentials. Hence, a new approach has been adopted for these potentials. First, we obtain the potential parameter spectrum (the set of values of the potential parameters that lead to an exact solution at a given energy). Then, the map that associates the parameter spectrum with the energy is inverted to give the energy spectrum for a given potential parameter. This procedure has been applied in 3D to obtain the energy spectrum for a special screened Coulomb potential with a barrier and in 1D for a single wave potential.

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Very few problems in quantum mechanics are exactly solvable. Despite the limited number of these problems, there are many advantages to obtaining their solutions using as many methods as possible. One such advantage is that these solutions give more insight into the physical concepts in quantum mechanics. Moreover, some of these potentials could be used as the unperturbed part of more realistic Hamiltonians. The search for exactly solvable potentials has been a subject of intense research since the advent of quantum mechanics. The factorization method [1], group theoretical techniques [2], super-symmetric quantum mechanics [3] and shape invariance [4] are few among many methods used to find exact solutions of the wave equation. In these developments, the main objective is to find solutions of the eigenvalue wave equation $H|\psi_n\rangle = E_n|\psi_n\rangle$, where H is the Hamiltonian, $\{E_n\}$ is the energy spectrum and $\{\psi_n\}$ are the corresponding state functions. Thus, one looks for a representation in which the Hamiltonian has a diagonal structure exhibiting the eigenvalues or the energy spectrum of the associated potential. In this respect, all attempts at enlarging the class of solvable potentials for which an exact solution is obtainable are important both theoretically and practically.

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In this communication, we search for an algebraic representation of the eigenvalue problem $H|\psi\rangle = E|\psi\rangle$ in the space of square integrable discrete basis with elements $\{\phi_n\}_{n=0}^{\infty}$. The main contribution of our work results from relaxing the usual restriction of a diagonal matrix representation of the wave operator. We only require that it be tridiagonal and symmetric (i.e., Jacobian) [5][5]. That is, the action of the wave operator on the elements of the basis is allowed to take the general form $(H - E)|\phi_n\rangle = a_n|\phi_n\rangle + b_{n-1}|\phi_{n-1}\rangle + b_n|\phi_{n+1}\rangle$, where a_n and b_n are the tridiagonal representation coefficients. We will show, by examples, that this tridiagonal representation approach will enlarge the class of solvable potentials and enable us to obtain new special solvable quantum problems, which cannot be obtained using the traditional diagonal representation approach. Moreover, in this work we also reintroduce the concept of a *parameter spectrum* where a solution of the problem is obtained at a single energy for a set (finite or infinite) of values of the potential parameters (the parameter spectrum) [6]. We start next by formulating the problem and introducing our approach for its solution.

The time-independent Schrödinger equation for a point particle of mass m in the field of a potential $V(x)$ is

$$\left[-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x) - E\right]\psi(x, E) = 0. \tag{1}$$

The physical configuration space coordinate belongs to the interval $x \in [x_-, x_+]$, which could be finite, infinite or semi-infinite. We make a transformation, $y = y(\lambda x)$, to a 'reference' configuration space with coordinate $y \in [-1, +1]$, where λ is a positive scale parameter. This transformation takes the wave equation (1) into

$$\left[(y')^2\frac{d^2}{\mathrm{d}y^2} + y''\frac{d}{\mathrm{d}y} - U(y) + \varepsilon\right]\psi(y, \varepsilon) = 0; \qquad \varepsilon = E/E_0, \qquad U = V/E_0, \tag{2}$$

where $E_0 = (\lambda\hbar)^2/2m$ and the primes stand for the derivative with respect to λx. Our approach is an algebraic one where we study the solution of the problem by constructing infinite-dimensional Hermitian matrices for the wave operator. A complete square integrable basis that is compatible with this problem and carries a faithful description of the wavefunction $\psi(y, \varepsilon)$ has the following basis elements [7]

$$\phi_n(y) = A_n(1 + y)^{\alpha}(1 - y)^{\beta}P_n^{(\mu,\nu)}(y), \tag{3}$$

where $P_n^{(\mu,\nu)}(y)$ is the Jacobi polynomial of degree $n = 0, 1, 2, \ldots$ and A_n is a normalization constant. The real parameters μ and ν are larger than -1 whereas α and β are positive. We expand the wavefunction in the complete basis (3) as $\psi(y, \varepsilon) = \sum_{n=0}^{\infty} f_n(\varepsilon)\,\phi_n(y)$, where $\{f_n\}$ are the (Fourier) expansion coefficients. Therefore, a complete solution of the problem is obtained if all $\{f_n(\varepsilon)\}_{n=0}^{\infty}$ are determined. In the present setting, it might be sufficient to point out that completeness and square integrability will guarantee boundedness and absolute convergence of the series, $\psi(y, \varepsilon) = \sum_{n=0}^{\infty} f_n(\varepsilon)\,\phi_n(y)$.

It is only for a limited and special set of potential functions $U(y)$ and with a proper choice of coordinate transformation $y(\lambda x)$ that the matrix representation of the wave operator becomes tridiagonal (Jacobian) [4, 7, 8]. That is, if we define the wave operator $J(y)$ as $J = (y')^2\frac{\mathrm{d}^2}{\mathrm{d}y^2} + y''\frac{\mathrm{d}}{\mathrm{d}y} - U(y) + \varepsilon$, then its matrix elements $J_{nm} = \langle\phi_n|J|\phi_m\rangle = 0$ for all $|n - m| \geqslant 2$. As stated above, the reason why we concentrate on tridiagonality is to enlarge the representation space of solvable problems and take advantage of the many important mathematical results that associate Jacobi matrices with orthogonal polynomials. Inserting

[5] The reason why we concentrate on the tridiagonality is to take advantage of the flurry of important mathematical results that associate tridiagonal (Jacobi) matrices with orthogonal polynomials. On the other hand, the mathematics literature is rich with work on the spectral theory of Jacobi (tridiagonal) operators. See [5] for some recent examples.

the wavefunction expansion in the wave equation (2) will result in the following three-term recursion relation for the expansion coefficients

$$J_{n,n-1} f_{n-1} + J_{n,n} f_n + J_{n,n+1} f_{n+1} = 0. \tag{4}$$

With a proper normalization, the solution of this recursion relation is a polynomial in some physical quantity (e.g., the energy or potential parameter). This is a well-known result in the context of orthogonal polynomials [9]. In most cases that correspond to the new class of solvable potentials, the orthogonal polynomials associated with the resulting recursion relation do not belong to any of the known classic polynomials (e.g., the Hermite, Chebyshev, Laguerre, etc) [10]. Nonetheless, these new polynomials are completely defined by their recursion relations and initial seed values. The solutions obtained in this work are *exact* solutions in the sense that all objects needed to calculate the sought-after physical quantities (e.g., the energy spectrum, phase shift, wavefunction, etc) are known and could be given to all orders in a systematic manner. The accuracy in the values obtained for such quantities is limited only by the computing machine accuracy; no physical approximations are ever involved.

We start the implementation of the approach by considering an S-wave ($\ell = 0$) problem in 3D, where x stands for the radial coordinate and we take $y(\lambda r) = 1 - 2\,\mathrm{e}^{-\lambda r}$. Following the same scheme outlined in [7, 8], the action of the wave operator (2) on the basis elements (3) is calculated with the help of the formulae given in the appendix of [7]. The result is as follows:

$$\begin{aligned}\frac{1+y}{1-y} J|\phi_n\rangle = \Bigg[& -\alpha(2\beta+1) - n(n+\mu+\nu+1) - n\left(y + \frac{\nu-\mu}{2n+\mu+\nu}\right) \\ & \times \left(\frac{2\alpha-\nu-1}{1+y} + \frac{\mu-2\beta}{1-y}\right) + \alpha(\alpha-1)\frac{1-y}{1+y} + \beta^2\frac{1+y}{1-y} + \frac{1+y}{1-y}(U-\varepsilon)\Bigg]|\phi_n\rangle \\ & + 2\frac{(n+\mu)(n+\nu)}{2n+\mu+\nu}\left(\frac{2\alpha-\nu-1}{1+y} + \frac{\mu-2\beta}{1-y}\right)\frac{A_n}{A_{n-1}}|\phi_{n-1}\rangle. \end{aligned} \tag{5}$$

The recursion relation and orthogonality formula of the Jacobi polynomials show that the matrix representation for the wave-like operator $\langle\phi_n|\frac{1+y}{1-y}J|\phi_m\rangle$ becomes tridiagonal only in the following three cases:

$$2\alpha = \nu + 1, \quad 2\beta = \mu: \qquad \frac{1+y}{1-y}U = -B\frac{1-y}{1+y} - Cy - D \tag{6a}$$

$$2\alpha = \nu + 1, \quad 2\beta = \mu + 1: \qquad \frac{1+y}{1-y}U = -B\frac{1-y}{1+y} - C\frac{y}{1-y} - \frac{D}{1-y} \tag{6b}$$

$$2\alpha = \nu + 2, \quad 2\beta = \mu: \qquad \frac{1+y}{1-y}U = -C\frac{y}{1+y} - \frac{D}{1+y}, \tag{6c}$$

where B, C and D are real dimensionless physical parameters and the basis parameters α and β must satisfy $\alpha(\alpha-1) = B$ and $\beta^2 = \varepsilon$. The second and third cases above correspond to the 'generalized Hulthén' potential, which is the sum of the Hulthén potential [11] and its square whose exact solutions are already known and have been obtained and classified by many researchers. Thus, we will not consider these two cases. On the other hand, the first case results in the following four-parameter potential function

$$U(r) = \frac{D}{\mathrm{e}^{\lambda r} - 1} + \frac{B}{(\mathrm{e}^{\lambda r} - 1)^2} + C\frac{1 - 2\,\mathrm{e}^{-\lambda r}}{\mathrm{e}^{\lambda r} - 1}. \tag{7}$$

The basis parameters in (3) become $\mu^2 = -4\varepsilon$, $\nu^2 = 4B+1$, $\alpha = (\nu+1)/2$ and $\beta = \mu/2$. Thus, real solutions of this problem in our approach are confined to negative energy (bound states)

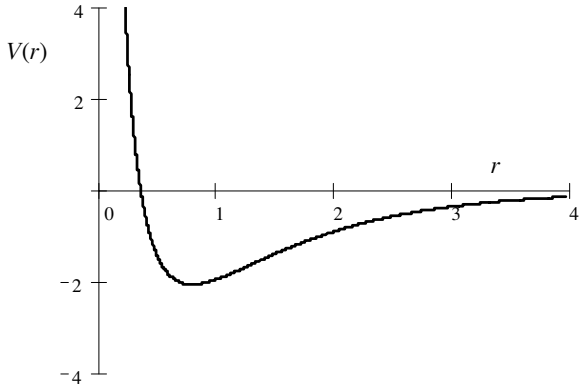

Figure 1. The 3D potential function (9) (in units of E_0) versus the radial coordinate (in units of λ^{-1}) for $\gamma = 0.7$ and $C = -5.0$.

and for $B \geqslant -\frac{1}{4}$. Moreover, the basis (3) becomes energy dependent through the parameter $\mu(\varepsilon)$. Using the recursion relation and orthogonality formula of the Jacobi polynomials [10], we obtain the following matrix elements of the wave-like operator

$$\left(\frac{1+y}{1-y}J\right)_{nm} = \left[D - B + \left(n + \frac{\nu+1}{2}\right)\left(n + \mu + \frac{\nu+1}{2}\right)\right]\delta_{nm} + C\langle n|y|m\rangle, \tag{8}$$

where the elements of the tridiagonal matrix $\langle n|y|m\rangle$ have already been evaluated previously in [8]. Now, the first two components of the potential (7) are well known. They make up the generalized Hulthén potential, which has an exact conventional (diagonal representation) solution. However, the last component is new and is the only component that contributes to the off-diagonal elements of the representation as shown in (8). Thus, the conventional diagonal representation requires that $C = 0$ washing out any hope to obtain a solution for the new potential component, if it existed. To make the new component of the solvable potential (the C-term in (7)) more interesting we limit our study to the choice $B = 0$ (hence, $\nu = +1$) and combine the first and last terms in (7) into the following short-range three-parameter potential

$$V(r) = V_0\frac{\mathrm{e}^{-\lambda r} - \gamma}{\mathrm{e}^{\lambda r} - 1}, \tag{9}$$

where $V_0 = -2E_0C$ is the potential strength and the dimensionless parameter γ is defined as $\gamma = \frac{1}{2}(1 + D/C)$ for $C \neq 0$. The basis functions (3) become

$$\phi_n(r) = A_n^{\mu}\,\mathrm{e}^{-\frac{1}{2}\lambda\mu r}(1 - \mathrm{e}^{-\lambda r})P_n^{(\mu,1)}(1 - 2\,\mathrm{e}^{-\lambda r}). \tag{10}$$

The potential (9) is a short-range potential with $1/r$ singularity at the origin. It is also interesting to note that, at short range and with $0 < \gamma < 1$, there is a clear resemblance of this potential with $V_0 > 0$ (see figure 1) to the attractive Coulomb potential with non-zero angular momentum. However, the potential valley here is not due to the centrifugal force attributed to the angular momentum. Moreover, the long-range behavior is not the same. Thus, in contrast to the Coulomb potential, we expect that the number of bound states is finite. Substituting from (8) into (4) results in the following three-term recursion relation for the expansion coefficients of the wavefunction

$$(2\gamma - 1)f_n = (d_n - a_n/C)f_n - b_{n-1}f_{n-1} - b_n f_{n+1}, \tag{11a}$$

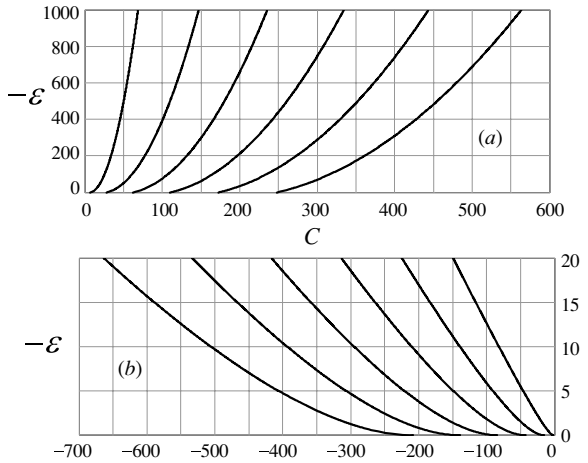

Figure 2. The energy spectrum associated with the 3D potential (9) as a function of the potential strength parameter and for $\gamma = \frac{1}{2}$.

where

$$\left.\begin{aligned} a_n(\varepsilon) &= (n+1)(n+\mu+1), \qquad d_n(\varepsilon) = \frac{\mu^2-1}{(2n+\mu+1)(2n+\mu+3)}, \\ b_n(\varepsilon) &= \frac{2}{2n+\mu+3}\sqrt{\frac{(n+1)(n+2)(n+\mu+1)(n+\mu+2)}{(2n+\mu+2)(2n+\mu+4)}}. \end{aligned}\right\} \tag{11b}$$

For space limitation, we study here the solution of this problem only for a fixed value of γ. To do that, we rewrite the recursion relation (11*a*) in terms of the new coefficients $g_n = \sqrt{a_n/a_0}\, f_n$, resulting in the following symmetrized recursion relation

$$C^{-1} g_n = \mathcal{A}_n g_n + \mathcal{B}_{n-1} g_{n-1} + \mathcal{B}_n g_{n+1}, \tag{12a}$$

where $\mathcal{A}_n = (d_n + 1 - 2\gamma)/a_n$ and $\mathcal{B}_n = -b_n/\sqrt{a_n\, a_{n+1}}$. The initial condition ($n = 0$) for this recursion is

$$C^{-1} g_0 = \mathcal{A}_0\, g_0 + \mathcal{B}_0\, g_1. \tag{12b}$$

We can also write (12*a*) as the eigenvalue equation $T_\gamma|g\rangle = C^{-1}|g\rangle$, where T_γ is the Jacobi matrix $(T_\gamma)_{nm} = \mathcal{A}_n\delta_{nm} + \mathcal{B}_{n-1}\delta_{n,m+1} + \mathcal{B}_n\delta_{n,m-1}$. Thus, for a given negative energy ε (equivalently, μ) and parameter γ, this eigenvalue equation gives an infinite set of discrete values for the potential strength (the C-parameter spectrum). They correspond to the set of all problems with these potential strengths whose energy spectra contain the value ε. The new recursion coefficients $\mathcal{A}_n$ and $\mathcal{B}_n$ approach the limit of large n as n^{-2}. Thus, using (12*a*) to calculate the C-parameter spectrum gives a more rapidly convergent result than using (11*a*) to calculate the γ-parameter spectrum. Figure 2 shows the calculated potential strength for a fixed parameter γ and for all bound states in a properly chosen energy range. The figure is shown with C on the horizontal axis and $-\varepsilon$ on the vertical axis to make it more convenient to visualize the energy spectrum. Thus, a vertical line that crosses the C-axis at any chosen potential strength value, say C', intersects the curves at the energy spectrum corresponding to the potential with parameters C' and γ. Out of these values, the most interesting are those at zero energy (i.e., at the boundary of the energy spectrum). We designate this sub-subset by the symbol $\{\hat{C}_n(\gamma)\}$ and list some of these values in table 1 for $\gamma = 1/2$. At these critical values, the state experiences transition from bound to resonance or vice versa (similar phenomenon

Table 1. The critical potential strength (the *C-parameter spectrum* at zero energy) for the 3D potential (9). The dimensionless parameter γ was taken equal to 1/2.

n	$\hat{C}_n > 0$	$\hat{C}_n < 0$
0	6.797 495 1801	−2.163 333 5216
1	27.431 119 0630	−15.905 126 4948
2	61.808 218 6058	−43.403 354 7967
3	109.934 889 3285	−84.653 920 8555
4	171.811 727 6666	−139.655 195 1096
5	247.438 860 5502	−208.406 948 4124

was observed for the Yukawa potential [12]). It is evident form figure 2 that for a given potential strength C that lies in the range $\hat{C}_n < C < \hat{C}_{n+1}$ the system will have $n+1$ bound states. Therefore, the set $\{\hat{C}_n(\gamma)\}$ is very important for bound states number counting.

In the second example, we consider a 1D problem with $x \in [-\infty, +\infty]$ and apply the transformation $y(\lambda x) = \tanh(\lambda x)$ to the wave equation (1). Following the same tridiagonalization scheme, we obtain the following most general solvable potential (i.e., the potential that preserves the tridiagonal structure of the wave operator)

$$U(x) = A\tanh(\lambda x) + \frac{B}{\cosh^2(\lambda x)} + C\frac{\tanh(\lambda x)}{\cosh^2(\lambda x)}, \tag{13}$$

where $\{A, B, C\}$ are real potential parameters and the basis parameters in (3) become $\mu^2 + \nu^2 = -2\varepsilon$, $\mu^2 - \nu^2 = 2A$, $\alpha = \nu/2$ and $\beta = \mu/2$. Thus, real solutions of this problem are also confined to negative energies (bound states). Again, the basis becomes energy dependent via the parameters μ and ν. Now, the first two components of the potential (13) are well known. These make up the hyperbolic Rosen–Morse potential [13] that has an exact conventional (diagonal representation) solution. However, the last component is new and it is the only one that contributes to the off-diagonal structure. The conventional diagonal representation requires that $C = 0$ eliminating any possibility of searching for a solution of this new potential component, if it were to exist. We limit our investigation to the choice $A = 0$ (i.e., $\mu = \pm\nu$) and consider only the case $\mu = +\nu$. The case $\mu = -\nu$, which requires that $-1 < \mu < +1$ (i.e., $-1 < \varepsilon < 0$), will be included in a full paper to follow. Similar to what we did in our previous example, we add the second term in (13) to the new C-term giving the following 1D potential

$$V = V_0\frac{\tanh(\lambda x) + \gamma}{\cosh^2(\lambda x)}, \tag{14}$$

where $V_0 = E_0 C$ and $\gamma = B/C$ with $C \neq 0$. Physically, the most interesting situation is when the parameter γ lies between -1 and $+1$ in which case the shape of this potential becomes a deformed single wave about the origin (see figure 3). However, if $|\gamma| \geqslant 1$ then it becomes a potential well if $\gamma C < 0$ or a potential barrier if $\gamma C > 0$. Due to the negative energy requirement in this problem, our approach can still handle the former case but not the latter. The basis functions (3) for this problem become

$$\phi_n(x) = A_n^{\mu}(\cosh\lambda x)^{-\mu} C_n^{\mu+\frac{1}{2}}(\tanh\lambda x), \tag{15}$$

where $C_n^{\nu}(z)$ is the Gegenbauer ultra-spherical polynomial [10]. The tridiagonal (Jacobi) matrix representation of the wave operator in this basis results in the following recursion relation for the expansion coefficients of the wavefunction

$$-\gamma f_n = (a_n/C) f_n + b_{n-1} f_{n-1} + b_n f_{n+1}, \tag{16}$$

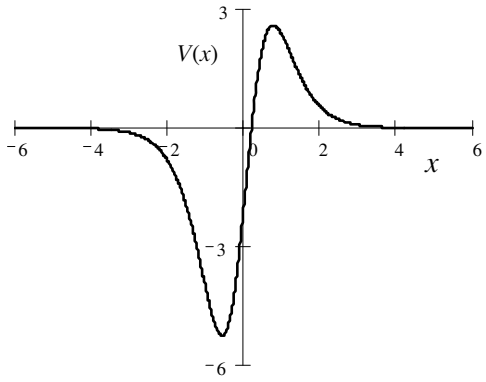

Figure 3. The 1D potential function (14) (in units of E_0) versus the x-coordinate (in units of λ^{-1}) for $\gamma = -0.2$ and $C = 10$.

Table 2. The critical potential strength for the 1D potential (14). The parameter γ was taken to be equal to $-1/5$.

n	$\hat{C}_n > 0$	$\hat{C}_n < 0$
0	0	−1.313 273 6608
1	9.448 622 3413	−23.953 744 7749
2	27.835 239 4121	−68.500 323 7805
3	55.366 792 7944	−135.304 755 2026
4	92.067 248 7219	−224.374 328 5307
5	137.940 353 0833	−335.710 317 4125

where $a_n = (n+\mu)(n+\mu+1)$ and $b_n = \frac{1}{2}\sqrt{\frac{(n+1)(n+2\mu+1)}{(n+\mu+1)^2-1/4}}$. The initial conditions are set so that $f_0 = 1$ and $f_{-1} = 0$. From here on, we proceed in a manner parallel to what was done above for the recursion relation (11*a*) in the 3D case. That is, we rewrite (16) in terms of the new polynomials $g_n = \sqrt{a_n/a_0}\, f_n$ giving the eigenvalue equation $T_\gamma|g\rangle = C^{-1}|g\rangle$, where $(T_\gamma)_{nm} = -(\gamma/a_n)\delta_{n,m} + B_{n-1}\delta_{n,m+1} + B_n\delta_{n,m-1}$ and $B_n = -b_n/\sqrt{a_n a_{n+1}}$. This equation gives the potential strength parameter spectrum for a given γ and a chosen negative energy. Table 2 shows the critical potential strength parameters $\{\hat{C}_n\}$ for $\gamma = -1/5$. It is interesting to note that there is no minimum critical potential strength for this 1D problem if $V_0 > 0$. By inverting the *C-parameter spectrum* map, we obtain the *energy spectrum* as shown in figure 4.

In conclusion, we have succeeded in finding two new solvable potentials in one and three dimensions by working in a complete square integrable basis that carries a tridiagonal (Jacobi) matrix representation for the wave operator. Consequently, the wave equation becomes equivalent to a three-term recursion relation for the expansion coefficients of the wavefunction in the basis. Finding solutions of the recursion relation is equivalent to solving the original problem. This method gives a larger class of solvable potentials. The usual diagonal representation constraint results in a reduction to the conventional class of solvable potentials. The details of our approach to the solution of these two problems will be presented in full papers following this paper.

The notion of *exact solvability* was defined in our present work as the ability to calculate all physical quantities in the problem to any desired accuracy limited only by the computing

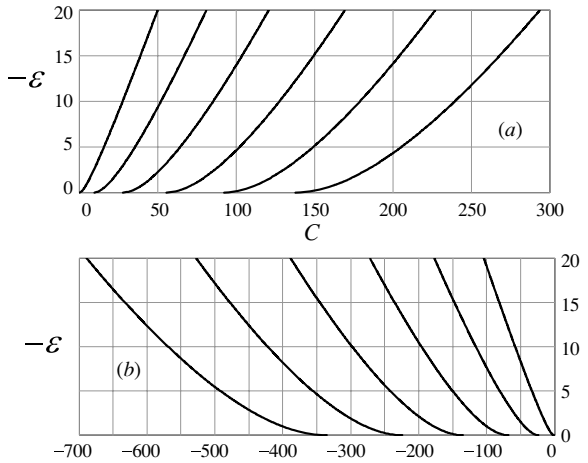

Figure 4. The energy spectrum associated with the 1D potential (14) as a function of the potential strength parameter and for $\gamma = -\frac{1}{5}$.

machine precision; no physical approximations are invoked. The difference between our tridiagonalization approach and the direct numerical integration approach is that we obtain a closed form for the wavefunction expansion coefficients resulting from the three-term recursion relation. We have also managed to have these expansion coefficients decrease fast enough asymptotically to ensure a controllable numerical accuracy. However, our solution strategy here differs from that of our previous work [7, 8] in that the basis is energy dependent. Thus, we had first to obtain the 'potential parameter spectrum' (i.e., the set of values of the potential parameters that leads to an exact solution for a given energy). We found that the *parameter spectrum* for the two potentials discussed in this work is infinite. Moreover, the map that associates the parameter spectrum with the energy is invertible which enabled us to obtain the *energy spectrum* very accurately. For the two problems, the energy spectrum is finite. In the present work, the above strategy was applied successfully to obtain a 3D S-wave solution for a special three-parameter $1/r$ singular but short-range potential with a barrier and for a single wave potential in 1D. We also gave an illuminating illustration of their energy and potential parameter spectra. Finally, we believe that the tridiagonal representation approach will enable us to enlarge the class of solvable quantum problems in all space dimensions and that it could easily be extended to non-central [14] as well as relativistic problems.

Acknowledgments

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